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Bounds on the crossing resolution of complete geometric graphs

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ABSTRACT

The *crossing resolution* of a geometric graph is the minimum crossing angle at which any two edges cross each other. In this paper, we present upper and lower bounds to the crossing resolution of the complete geometric graphs.

1. Introduction

A *geometric graph D* (also called a *straight-line drawing*) is a representation of a graph G in the plane with each vertex u represented as a distinct point p_u and each edge (u, v) represented as the open line segment (p_u, p_v), such that no edge intersects a vertex, and two edges intersect each other in at most one point. The *crossing resolution* $\phi(D)$ of a non-planar geometric graph D is the minimum angle at which any two edges cross. Given a graph G (a drawing D) we denote by $V(G)$ and $E(G)$ ($V(D)$ and $E(D)$) the set of vertices and the set of edges of G (of D), respectively.

The crossing resolution of a non-planar geometric graph is at most $\frac{\pi}{2}$. A recent paper [4] proves that the non-planar geometric graphs with n vertices having crossing resolution $\frac{\pi}{2}$ have at most $4n - 10$ edges, which is a tight upper bound. Therefore, the requirement of having straight-line edges must be relaxed if one wishes to draw non-sparse graphs with crossing resolution $\frac{\pi}{2}$. In the same paper, it is indeed proved that every non-planar graph has a drawing with crossing resolution $\frac{\pi}{2}$ and three bends per edge (a *bend* is a point common to two consecutive segments of an edge drawn as a polyline). It is also shown that if a drawing with orthogonal crossings has at most one or two bends per edge, then its number of edges must be $O(n^{\frac{4}{3}})$ and $O(n^{\frac{7}{4}})$, respectively. These last upper bounds were recently improved by Arikushi et al. [2] who showed that if a graph admits a drawing with at most one bend (two bends) per edge and orthogonal crossings, then this graph has at most $6.5n - 13$ $(74.2n)$ edges. Dujmović et al. [8] study non-planar geometric graphs with crossing resolution other than $\frac{\pi}{2}$ and prove that for any fixed α such that $0 < \alpha < \frac{\pi}{2}$, the geometric graphs with crossing resolution α are sparse. Those complete bipartite graphs that admit a straight-line drawing with crossing resolution $\frac{\pi}{2}$ are characterized in [5]. Finally, tradeoffs between crossing resolution and other typical graph drawing constraints, including area requirement and edge upwardness for drawings of directed acyclic graphs, were studied in [1].

In this paper, we investigate the problem of computing drawings of non-planar graphs such that the crossing resolution is as large as possible. We recall that if one bend per edge is allowed, then every graph admits a drawing whose crossing resolution is at least $\frac{\pi}{2} - \epsilon$ for any arbitrarily chosen $\epsilon > 0$ [6]. We focus here on drawings with straight-line edges and

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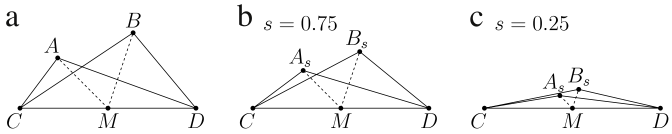

Fig. 1. Illustration of Lemma 1. (a) The two triangles ACD and BCD. (b) The two triangles A_sCD and B_sCD for $s = 0.75$. A_s is outside the triangle B_sCD. (c) The two triangles A_sCD and B_sCD for $s = 0.25$. A_s is inside the triangle B_sCD.

establish upper and lower bounds on the crossing resolution of complete geometric graphs. A *complete geometric graph* D_n is a straight-line drawing of the complete graph K_n for some $n > 1$. We prove the two following main results.

Theorem 1. *For any given $\epsilon > 0$ and $n > 4$, there exists a complete geometric graph D_n such that $\phi(D_n) \geq \frac{\pi}{\lceil \frac{n}{3} \rceil} - \epsilon$.*

Theorem 2. *Let D_n be a complete geometric graph with $n \geq 12$ vertices. Then $\phi(D_n) \leq \frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1}$.*

We remark that the study of the crossing resolution of drawings of graphs, besides of its theoretical interest, has recently received considerable attention due to results in the fields of Human–Computer Interaction and Cognitive Perception showing that a drawing of a graph is easier for humans to understand when the crossing angles are large; see, e.g., [9–11]. We also observe that our research is related to the study of geometric graphs using a limited set of edge slopes. Indeed, the crossing resolution of geometric graph is at least the minimum angle formed by any two edges with different slopes. Although the number of distinct edge slopes required by a straight-line drawing of a graph can be arbitrarily high [3,14], there are specific graph families for which a limited number of edge slopes suffice (see, e.g., [12,13]).

The remainder of the paper is organized as follows. Section 2 gives a lower bound on the crossing resolution of complete geometric graphs, and Section 3 gives upper bounds. Conclusions and open problems can be found in Section 4.

2. Lower bound

We start with a geometric lemma that will be used to prove our lower bound. We denote with x_P, y_P the x- and y-coordinate of a point P.

Lemma 1. *Let ACD and BCD be two triangles with common base CD, such that $x_C < x_A < x_B < x_D$ and $y_C = y_D < y_A < y_B$. Let M be the midpoint of the line segment CD. For $0 < s \leq 1$, let $A_s = sA + (1 - s)M$ and $B_s = sB + (1 - s)M$. There exists a real number $0 < s_0 \leq 1$ such that for every $s \leq s_0$ point A_s is inside the triangle B_sCD.*

Proof. The conditions of the lemma are illustrated in Fig. 1(a). Note that for all s, the line segment A_sB_s is parallel to the line segment AB; thus the slope of A_sB_s is constant for all s. However, as s decreases, the slope of CA_s decreases, and for s close to zero, the angle DCA_s is close to zero (see Fig. 1(b) and (c)). Thus as long as s is small enough, the slope of CA_s is less than the slope of A_sB_s. This ensures that A_s is inside the triangle B_sCD. □

Next, we prove a lemma that establishes an initial lower bound on the minimum crossing angle of a complete geometric graph which will be improved by the main theorem of this section. This lower bound is needed in the proof of Lemma 3, which is the foundation for proving Theorem 1.

Lemma 2. *Let D_n be a complete geometric graph with $n > 4$ vertices such that the vertices of D_n coincide with the vertices of a regular n-gon. Then, $\phi(D_n) = \frac{2\pi}{n}$.*

Proof. Enumerate the vertices of D_n from 0 to $n - 1$ counterclockwise and assume, without loss of generality, that the line through vertices 0 and 1 coincides with the x-axis. Note that, for any pair of indices $i, j \in \{0, 1, \ldots, n - 1\}$ such that $j > i$, the line segment oriented from i to j forms an angle of value $(i + j - 1)\frac{\pi}{n}$ with the positive x-axis. If $0 \leq i < k < j < \ell < n$ then the straight-line segment from i to j crosses the straight-line segment from k to ℓ, and the crossing angles formed by these two segments have values $(k + \ell - i - j)\frac{\pi}{n}$ and $(n - (k + \ell - i - j))\frac{\pi}{n}$. Since $\ell - j \geq 1$ and $k - i \geq 1$, we have $(k + \ell - i - j)\frac{\pi}{n} \geq \frac{2\pi}{n}$. Also, since $\ell < n - 1$ and $i \geq 0$, we have $(n - (k + \ell - i - j))\frac{\pi}{n} \geq \frac{2\pi}{n}$.

Finally, for $k = i + 1$, $j = i + 2$, and $\ell = i + 3$, we have $(k + \ell - i - j)\frac{\pi}{n} = \frac{2\pi}{n}$. □

Observe that the crossing resolution of Lemma 2 is basically the best possible one can achieve if all vertices of K_n are in convex position. Indeed, every convex straight-line drawing of K_n has $\lfloor \frac{n}{2} \rfloor$ pairwise crossing edges; thus its crossing resolution is at most $\frac{\pi}{\lfloor \frac{n}{2} \rfloor}$.

Now a general lemma is proved, directly leading to a proof of Theorem 1.

Lemma 3. *Let n, a, b be three positive integers such that $n = a + b > 4$, a is even, and $a > b$. Then, for any given $\epsilon > 0$, there exists a complete geometric graph D_n such that $\phi(D_n) \geq \frac{2\pi}{\max(a, 2b)} - \epsilon$.*

Proof. We partition the vertex set of K_n into V_{red} and V_{blue} with $|V_{red}| = a$ and $|V_{blue}| = b$, and the edge set of K_n into three sets: (i) $E_{red} = V_{red} \times V_{red} \setminus \{(v, v) \mid v \in V_{red}\}$; (ii) $E_{blue} = V_{blue} \times V_{blue} \setminus \{(v, v) \mid v \in V_{blue}\}$; (iii) $E_{green} = V_{red} \times V_{blue}$. Denote the graphs (V_{red}, E_{red}) and (V_{blue}, E_{blue}) by K_a^{red} and K_b^{blue}, respectively.

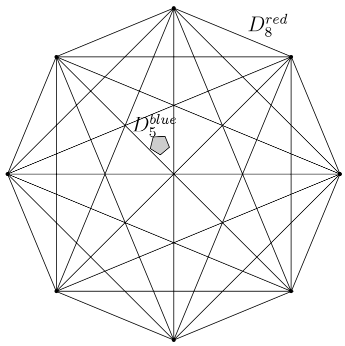

Fig. 2. An illustration of the construction of Lemma 3 for K_{13}. In this case $a = 8$ and $b = 5$.

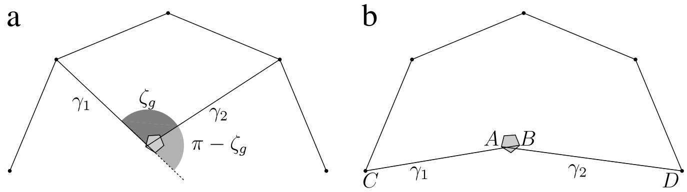

Fig. 3. (a) A crossing between two green edges γ_1 and γ_2. The number of edges k between the red vertices of γ_1 and γ_2 is 2. (b) A crossing between two green edges γ_1 and γ_2. The number of edges k between the red edges of γ_1 and γ_2 is $\frac{a}{2} = 4$.

We construct a drawing D_n of K_n. The subgraph K_a^{red} is drawn as a regular a-gon centered at the origin; we denote this drawing by D_a^{red}. Note that, since a is even, there are $\frac{a}{2}$ red edges that meet at the origin, and that D_a^{red} has a faces that are incident to the origin. We choose one of these faces, and draw K_b^{blue} as a regular b-gon strictly inside it. We then rotate this regular b-gon so that no edge of E_{blue} is parallel to any edge of E_{red}; denote this drawing of K_b^{blue} by $D_b^{blue}(1)$ (see Fig. 2 for an illustration). Now, for any real number s such that $0 < s \leq 1$, we denote by $D_b^{blue}(s)$ the drawing of K_b^{blue} formed by shrinking $D_b^{blue}(1)$ by a factor of s (that is, the point p of $D_b^{blue}(1)$ becomes sp in $D_b^{blue}(s)$). Observe that $D_b^{blue}(s)$ lies strictly inside a face of D_a^{red}. Thus, there are no crossings between blue edges and red edges.

We now prove that, for any $\epsilon > 0$, there exists a sufficiently small value of s such that $\phi(D_n) \geq \frac{2\pi}{\max(a,2b)} - \epsilon$.

Consider crossing angles formed by two red edges or by two blue edges, that is, consider the minimum crossing angles $\phi(D_a^{red})$ and $\phi(D_b^{blue}(s))$ (for any s). By Lemma 2, we have:

$$\phi(D_a^{red}) = \frac{2\pi}{a} \geq \frac{2\pi}{\max(a, 2b)}, \tag{1}$$

$$\phi(D_b^{blue}(s)) = \frac{2\pi}{b} > \frac{2\pi}{\max(a, 2b)}. \tag{2}$$

It follows that the crossings that do not involve the green edges satisfy the statement. We now consider crossings that involve green edges.

First consider a crossing between two green edges γ_1 and γ_2 (see Fig. 3(a)), and let u_1 and u_2 be the red vertices of γ_1 and γ_2, respectively. Vertices u_1 and u_2 are connected by two paths along the boundary of the convex hull of $V(D_a^{red})$. Let k be the number of edges in the shortest of these two paths. Note that k is an integer such that $0 \leq k \leq \frac{a}{2}$. If $k = 0$ then γ_1 and γ_2 do not cross. Hence assume that $k > 0$. There are two crossing angles ζ_g and $\pi - \zeta_g$ between γ_1 and γ_2. If s is sufficiently small, then the endpoints of γ_1 and γ_2 in $D_b^{blue}(s)$ are very close to the origin, and we can ensure that ζ_g is arbitrarily close to $\frac{2k\pi}{a}$. More precisely, for any $\epsilon > 0$ there exists a sufficiently small value of s such that:

$$\zeta_g > \frac{2k\pi}{a} - \epsilon. \tag{3}$$

With the same argument, there exists a sufficiently small value of s such that:

$$\pi - \zeta_g > \frac{2\left(\frac{a}{2} - k\right)\pi}{a} - \epsilon. \tag{4}$$

Hence, there exists a sufficiently small value of s that guarantees both (3) and (4). Since $k \geq 1$, we have $\zeta_g > \frac{2\pi}{a} - \epsilon$. If $k \neq \frac{a}{2}$, then $\frac{a}{2} - k > 1$ and therefore $\pi - \zeta_g > \frac{2\pi}{a} - \epsilon$. Thus, if $k \neq \frac{a}{2}$ then the crossing angles between γ_1 and γ_2 satisfy

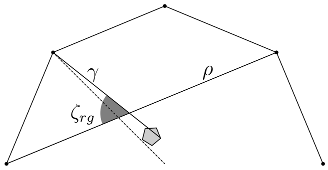

Fig. 4. A crossing between a green edge γ and a red edge ρ.

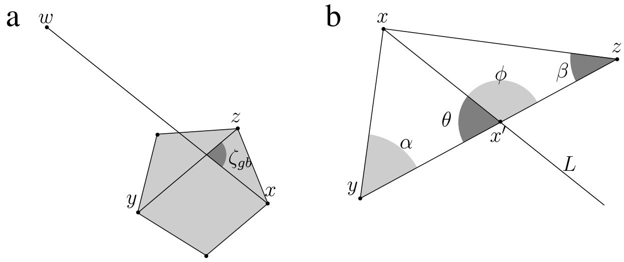

Fig. 5. (a) A crossing between a green edge and a blue edge. (b) Illustration of a property of the triangles.

the statement. We now prove that if $k = \frac{a}{2}$ then there exists a sufficiently small value of s such that γ_1 and γ_2 do not cross. If $k = \frac{a}{2}$, then γ_1 and γ_2 meet the convex hull of D_a^{red} at points that are diametrically opposite each other, as in Fig. 3(b). Let the endpoints of γ_1 and γ_2 be C, B, A, D, as in Fig. 3(b). Assume, without loss of generality, that CD is horizontal. Note that AB is not horizontal, since no blue edge is parallel to a red edge. Thus we can assume (also without loss of generality) that the y coordinate of A is less than the y coordinate of B. By Lemma 1, and taking s small enough, we can ensure that A is inside the triangle BCD, and therefore γ_1 and γ_2 do not cross.

Next consider crossings between red edges and green edges (see Fig. 4). Let ζ_{rg} be a crossing angle between a red edge ρ and a green edge γ. As $s \to 0$, the endpoint of γ in $D_b^{blue}(s)$ approaches the origin of D_a^{red}, and then γ approximates a radius of D_a^{red}, without crossing it. This means that ζ_{rg} approximates the value of a crossing angle between two red edges, that is:

$$\zeta_{rg} > \frac{2\pi}{a} - \epsilon. \tag{5}$$

Finally consider crossings between blue edges and green edges (see Fig. 5(a)). For this case we need a simple geometric inequality for triangles (see Fig. 5(b)): if a line L from a vertex x of triangle xyz intersects the line segment (y, z) at an internal point x', then the angles $\theta = \angle xx'y$ and $\phi = \angle xx'z$ satisfy $\min(\theta, \phi) > \min(\alpha, \beta)$, where α and β are the angles of triangle xyz at y and z. Now, suppose that a green edge (x, w) crosses a blue edge (y, z), and let ζ_{gb} be the minimum crossing angle formed by (x, w) and (y, z). The simple geometric inequality implies that

$$\zeta_{gb} > \min(\angle xyz, \angle xzy).$$

Since the angle between two consecutive edges around a vertex of D_b^{blue} is at least $\frac{\pi}{b}$, it follows that

$$\zeta_{gb} > \frac{\pi}{b}. \tag{6}$$

This completes the proof. □

The next theorem shows that there exists a complete geometric graph D_n whose minimum crossing angle is guaranteed to be larger than, roughly, $\frac{3\pi}{n}$.

Theorem 1. *For any given $\epsilon > 0$ and $n > 4$ there exists a complete geometric graph D_n such that $\phi(D_n) \geq \frac{\pi}{\lceil \frac{n}{3} \rceil} - \epsilon$.*

Proof. First suppose that $n \equiv 0\ (mod\ 3)$. In this case, take $a = \frac{2n}{3}$ and $b = \frac{n}{3}$ and the theorem follows from Lemma 3.

Next suppose that $n \equiv 1\ (mod\ 3)$; take $a = \lfloor \frac{2n}{3} \rfloor$ and $b = \lceil \frac{n}{3} \rceil$. Note that a is even and $a < 2b$; thus again the result follows from Lemma 3.

Finally, if $n \equiv 2\ (mod\ 3)$, then $n + 1 \equiv 0\ (mod\ 3)$, and we can draw K_{n+1} with a minimum crossing angle of $\frac{2\pi}{\frac{(n+1)}{3}} - \epsilon = \frac{2\pi}{\lceil \frac{n}{3} \rceil} - \epsilon$; deleting a vertex gives a drawing of K_n that satisfies the inequality in Theorem 1. □

Fig. 6 shows an application of Theorem 1 to K_6. The crossing resolution in the figure can be made arbitrarily close to $\frac{\pi}{2}$ by shrinking the segment AB and moving it toward the center O, which matches the lower bound established by Theorem 1. Note that K_6 does not admit a straight-line drawing with crossing resolution $\frac{\pi}{2}$ because its number of edges is larger than $4n - 10$, which is an upper bound for the geometric graphs having crossing resolution $\frac{\pi}{2}$ [4].

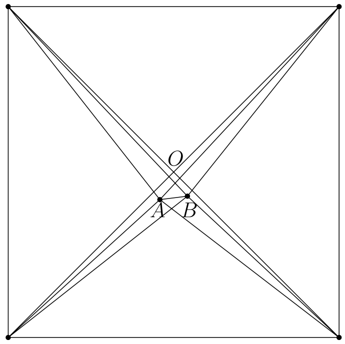

Fig. 6. A complete geometric graph D_6 with crossing resolution $\frac{\pi}{2} - \epsilon$.

3. Upper bounds

In this section we provide three different upper bounds on the crossing resolution of complete geometric graphs. We also give examples where each of these upper bounds is tighter than the other two.

The *geometric thickness* of a graph G, denoted as $\overline{\theta}(G)$, is the minimum value of k such that G has a straight-line drawing in which the edges can be colored in k colors so that no two edges of the same color cross. The following theorem establishes an upper bound to the crossing resolution $\phi(D)$ of a geometric graph D in terms of the geometric thickness of the (combinatorial) graph isomorphic to D.

Lemma 4. *Let D be a non-planar straight-line drawing of a graph G. Then $\phi(D) \leq \frac{\pi}{\overline{\theta}(G)-1}$.*

Proof. Partition the edges of D into buckets $B_1, B_2, \ldots, B_k$ where $k = \lceil \frac{\pi}{\phi(D)} \rceil$, as follows. For $j = 1, 2, \ldots, k-1$, let B_j be the set of edges whose direction s satisfies $(j-1)\phi(D) \leq s < j\phi(D)$. The *direction* of an edge e is defined as the angle swept from the x-axis in an anticlockwise direction to the straight line containing e. The direction is an angle in the range $[0, \pi)$. Let B_k be the set of edges whose direction s satisfies $(k-1)\phi(D) \leq s < \pi$. The difference between directions of edges in the same bucket is strictly less than $\phi(D)$; thus two edges in the same bucket cannot cross. Hence, by definition, $\overline{\theta}(G) \leq k$. The statement follows by observing that $\overline{\theta}(G) \leq \frac{\pi}{\phi(D)} + 1$. $\square$

Dillencourt et al. [7] proved that $\overline{\theta}(K_n) \geq \lceil \frac{n}{5.646} + 0.342 \rceil$ for $n \geq 12$. An immediate consequence of this result and of Lemma 4 is the following theorem.

Theorem 2. *Let D_n be any complete geometric graph with $n \geq 12$ vertices. Then $\phi(D_n) \leq \frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1}$.*

3.1. Further upper bounds

In this subsection we present two additional upper bounds on the crossing resolution of a complete geometric graph D_n in terms of two parameters, other than the total number n of vertices: (i) the number n' of vertices on the convex hull of $V(D_n)$; and (ii) the maximum number n'' of vertices in convex position.

Theorem 3. *Let D_n be a complete geometric graph with $n > 4$ vertices and let n' be the number of elements of $V(D_n)$ that belong to the convex hull of $V(D_n)$. Then $\phi(D_n) \leq \frac{6n-6-4n'}{n(n-1)-2n'}\pi$.*

Proof. The proof uses a technique similar to the one used by Dujmović et al. [8, Theorem 1]. Consider the geometric graph D'' obtained from D_n by deleting the edges on the convex hull; D'' has n vertices and $m'' = \frac{n(n-1)}{2} - n'$ edges. Partition the m'' edges of D'' into buckets $B_1, B_2, \ldots, B_k$ where $k = \lceil \frac{\pi}{\phi(D_n)} \rceil$, as follows. For $j = 1, 2, \ldots, k-1$, let B_j be the set of edges whose direction s satisfies $(j-1)\phi(D_n) \leq s < j\phi(D_n)$. Let B_k be the set of edges whose direction s satisfies $(k-1)\phi(D_n) \leq s < \pi$. As proved by Dujmović et al. [8, Theorem 1], there exists a rotation of the drawing D_n such that the last bucket B_k contains at most $\beta \cdot m''$ edges, where

$$\beta = \frac{\frac{\pi}{\phi(D_n)} - \left\lfloor \frac{\pi}{\phi(D_n)} \right\rfloor}{\frac{\pi}{\phi(D_n)}}.$$

The difference between directions of edges in B_j $(1 \leq j \leq k)$ is strictly less than $\phi(D_n)$; thus the subgraph formed by B_j is a planar graph and it remains planar even if we add to it the n' edges of the convex hull. Denote by G''_j the graph containing all the vertices of D_n, all the edges of B_j and all the n' edges of the convex hull ($1 \leq j \leq k-1$). Analogously denote by G''_k the graph containing all the vertices of D_n and all the edges of B_k. We have:

$$\sum_{j=1}^{k} |E(G''_j)| = m'' + (k-1)n', \tag{7}$$

because the sum of the edges in the buckets is m'' and we added n' edges to the first $k-1$ buckets.

On the other hand, since each G''_j ($1 \leq j \leq k-1$) is planar and has n' vertices on the external face, the number of edges of G''_j is at most $3n - 3 - n'$. Since G''_k has at most $\beta \cdot m''$ edges, we have:

$$\sum_{j=1}^{k} |E(G''_j)| \leq (k-1)(3n - 3 - n') + \beta \cdot m''. \tag{8}$$

Combining Eqs. (7) and (8) we obtain:

$$m'' + (k-1)n' \leq (k-1)(3n - 3 - n') + \beta \cdot m''$$

which becomes:

$$\frac{1-\beta}{k-1} \leq \frac{3n-3-2n'}{m''}.$$

Since $k = \lceil \frac{\pi}{\phi(D_n)} \rceil$, we have $k - 1 \leq \lfloor \frac{\pi}{\phi(D_n)} \rfloor$ and therefore:

$$\frac{1-\beta}{\left\lfloor \frac{\pi}{\phi(D_n)} \right\rfloor} \leq \frac{1-\beta}{k-1} \leq \frac{3n-3-2n'}{m''}.$$

Furthermore, by the definition of β, $\frac{\phi(D_n)}{\pi} = \frac{1-\beta}{\lfloor \frac{\pi}{\phi(D_n)} \rfloor}$. Thus:

$$\frac{\phi(D_n)}{\pi} \leq \frac{3n-3-2n'}{m''}$$

and

$$\phi(D_n) \leq \frac{3n-3-2n'}{m''}\pi.$$

The statement follows from $m'' = \frac{n(n-1)}{2} - n'$. □

Theorem 4. *Let D_n be a complete geometric graph with $n > 4$ vertices and let n'' be the maximum number of elements of $V(D_n)$ that are in convex position. Then $\phi(D_n) \leq \frac{2\pi}{n''}$.*

Proof. The complete geometric graph $D_{n''}$ consisting of the n'' points in convex position and of the edges between them has all the vertices in convex position. Then an upper bound on $\phi(D_{n''})$ derives from the one in Lemma 1 with $n = n' = n''$, i.e., $\phi(D_n) \leq \phi(D_{n''}) \leq \frac{6n''-6-4n''}{n''(n''-1)-2n''}\pi = \frac{2(n''-3)}{n''(n''-3)}\pi = \frac{2\pi}{n''}$. □

Theorems 2–4 imply that the upper bound on $\phi(D_n)$ is the minimum between three different terms. Depending on the values of n' and n'' one of the three terms results in a better bound for $\phi(D_n)$. In what follows we show three different types of complete geometric graphs where each of the three terms gives the better bound. In all cases the vertices are drawn on a set of concentric circles $C_1, C_2, \ldots, C_k$ ($k \geq 2$). We assume that circle C_i has radius smaller than circle C_{i-1} for $i = 2, \ldots, k$.

Example 1. Let $n = 4h$, with $h \geq 29$ and h is an integer. Consider a complete geometric graph D_n constructed as follows. The vertices are drawn on five concentric circles. The outermost circle C_1 contains 3 vertices; the circles C_2, C_3, and C_4 contain $\frac{n}{4} - 1$ vertices each; circle C_5 contains $\frac{n}{4}$ vertices. Also, the vertices are arranged on the circle so that the convex hull of $V(D_n)$ is the triangle defined by the vertices on C_1. In this case $n' = 3$ and $n'' = \frac{n}{4}$. Thus:

$$\frac{2\pi}{n''} = \frac{8\pi}{n}$$
$$\frac{6n-6-4n'}{n(n-1)-2n'}\pi = \frac{6n-18}{n(n-1)-6}\pi \geq \frac{6n-18}{n^2}\pi.$$

We have:

$$\frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1} \leq \frac{\pi}{\frac{n}{5.646} + 0.342 - 1} = \frac{5.646\pi}{n - 3.715068}$$
$$\frac{8\pi}{n} > \frac{5.646\pi}{n - 3.715068} \quad \text{for } n > 13$$
$$\frac{6n - 18}{n^2}\pi > \frac{5.646\pi}{n - 3.715068} \quad \text{for } n > 113.$$

Since $n \geq 116$ (because $h \geq 29$) we have $\phi(D_n) \leq \frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1}$.

Example 2. Let $n = 2h$, with $h \geq 7$ and h is an integer. Consider a complete geometric graph D_n constructed as follows. The vertices are drawn on three concentric circles. The circles C_1 and C_2 contain $\frac{n}{2} - 1$ vertices each; the circle C_3 contains two vertices. Also, the vertices are arranged on the circle so that the convex hull of $V(D_n)$ is the polygon defined by the vertices on C_1. In this case $n' = n'' = \frac{n}{2} - 1$. Thus:

$$\frac{2\pi}{n''} = \frac{4\pi}{n - 2}$$
$$\frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1} > \frac{\pi}{\left(\frac{n}{5.646} + 0.342\right)} = \frac{5.646\pi}{n + 1.930932}.$$

We have:

$$\frac{6n - 6 - 4n'}{n(n-1) - 2n'}\pi = \frac{4n - 2}{n^2 - 2n + 2}\pi < \frac{4n}{n^2 - 2n + 2}\pi < \frac{4\pi}{n - 2}$$
$$\frac{4\pi}{n - 2} < \frac{5.646\pi}{n + 1.930932} \quad \text{for } n > 12.$$

Since $n \geq 14$ (because $h \geq 7$) we have $\phi(D_n) \leq \frac{6n-6-4n'}{n(n-1)-2n'}\pi$.

Example 3. Let $n = 3h$, with $h \geq 7$ and h is an integer. Consider a complete geometric graph D_n constructed as follows. The vertices are drawn on two concentric circles C_1 and C_2. The circle C_1 contains $\frac{2n}{3}$ vertices, while C_2 contains $\frac{n}{3}$ vertices. Also, the vertices are arranged on the circle so that the convex hull of $V(D_n)$ is the polygon defined by the vertices on C_1. In this case $n' = n'' = \frac{2n}{3}$. Thus:

$$\frac{2\pi}{n''} = \frac{3\pi}{n}$$
$$\frac{6n - 6 - 4n'}{n(n-1) - 2n'}\pi = \frac{10n - 18}{n(3n - 7)}\pi > \frac{10n - 18}{3n^2}\pi.$$

Since $n > 18$ (because $h \geq 7$), we have $\frac{10n-18}{3n^2}\pi > \frac{9n}{3n^2}\pi = \frac{3\pi}{n}$. Furthermore:

$$\frac{\pi}{\lceil \frac{n}{5.646} + 0.342 \rceil - 1} > \frac{5.646\pi}{n + 1.930932}$$

we have $\frac{5.646\pi}{n+1.930932} > \frac{3}{n}$ for $n > 3$; since $n \geq 21$ (because $h \geq 7$) it results $\phi(D_n) \leq \frac{2\pi}{n''}$.

4. Conclusions and open problems

This paper initiates the study of the crossing resolution of dense geometric graphs. It proves upper and lower bounds on the crossing resolution of complete geometric graphs. We conclude this paper by mentioning two research directions that in our opinion would be worth to investigate.

The first research direction is to reduce (and possibly close) the gap between upper and lower bounds established in Theorems 1 and 2. We have some evidence that the gap can be closed for small values of n. For example, Fig. 7 shows that there exists a complete geometric graph D_7 with crossing resolution arbitrarily close to $\frac{\pi}{2}$. It is possible to increase the crossing resolution by shrinking the size of the triangle ABC and moving it toward the center O. Note that, by Theorem 1 of [4], there cannot be a straight-line drawing of K_7 with crossing resolution $\frac{\pi}{2}$. We conjecture that the gap between upper and lower bounds can be arbitrarily reduced also for other small values of n. For example, we believe that the upper bound to the crossing resolution is $\frac{\pi}{3}$ for any straight-line drawing of K_8 and K_9, while it is $\frac{\pi}{4}$ for K_{10}.

Another interesting research direction is to establish upper and lower bounds on the crossing resolution of other families of dense geometric graphs. For example, it would be interesting to study complete bipartite geometric graphs. Related to

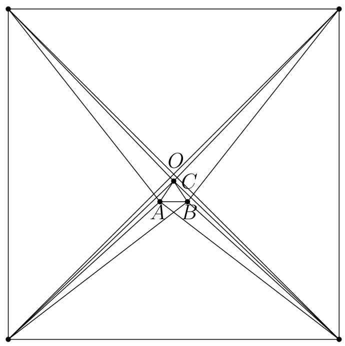

Fig. 7. A complete geometric graph D_7 with crossing resolution $\frac{\pi}{2} - \epsilon$.

this, it may be worth mentioning that a complete bipartite graph K_{n_1,n_2} with $n_1 \leq n_2$ admits a straight-line drawing with crossing resolution $\frac{\pi}{2}$ if and only if either $n_1 \leq 2$, or $n_1 \leq 3$ and $n_2 \leq 4$ [5].

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